

Gravothermal oscillations in multi-component models of star clusters

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ABSTRACT

In this paper, gravothermal oscillations are investigated in multi-component star clusters which have power law initial mass functions (IMF). For the power law IMFs, the minimum masses (m_{min}) were fixed and three different maximum stellar masses (m_{max}) were used along with different power-law exponents (α) ranging from 0 to -2.35 (Salpeter). The critical number of stars at which gravothermal oscillations first appear with increasing N was found using the multi-component gas code SPEDI. The total mass (M_{tot}) is seen to give an approximate stability condition for power law IMFs with fixed values of m_{max} and m_{min} independent of α . The value $M_{tot}/m_{max} \simeq 12000$ is shown to give an approximate stability condition which is also independent of m_{max} , though the critical value is somewhat higher for the steepest IMF that was studied. For appropriately chosen cases, direct N-body runs were carried out in order to check the results obtained from SPEDI. Finally, evidence of the gravothermal nature of the oscillations found in the N-body runs is presented.

Key words: globular clusters: general; methods: numerical; methods: n-body simulations.

1 INTRODUCTION

The condition for the onset of gravothermal oscillations is best understood for the case of one-component star clusters, clusters consisting of stars of equal mass. Goodman (1987) found that gravothermal oscillations first appear when the number of stars N is greater than 7000. This condition has also been confirmed with Fokker-Planck calculations (Cohn et al 1989) and by direct N-body simulations (Makino 1996). However, the multi-component case is more complicated. This is due to the fact that the presence of several components introduces new dynamical processes into the system, and several additional parameters in addition to N .

Even for the two-component case, which is the simplest kind of mass spectrum, the condition for the onset of gravothermal oscillations is not so simple. Two-component models can be subdivided into Spitzer stable and Spitzer unstable cases depending on whether or not the two components can achieve equipartition of kinetic energy during core collapse (Spitzer 1987). Kim, Lee & Goodman (1998) studied a range of Spitzer stable two-component models. Their research supported the applicability of the Goodman stability parameter ϵ (see Goodman 1993) as a stability criterion.

Breen & Heggie (2012), whose research focused on the more general Spitzer unstable two-component case, indicated that the occurrence of gravothermal instability depends approximately on the number of stars in the heavier component. Breen & Heggie (2012) also found that the critical value of ϵ depended on the parameters of the mass function (e.g. stellar mass ratio). However, by using a slightly modified version of ϵ , one with a modified definition of the half mass relaxation time, they found a nearly constant critical value.

Murphy et al (1990) found that the post-collapse evolution of multi-component models was stable to much higher values of N than in one-component models and that the value of N at which gravothermal oscillations appeared varied with different mass functions. They studied seven-component systems constructed to approximate evolved power law IMFs with masses ranging from 0.1 to $1.2M_{\odot}$. The power law exponent that they considered ranged from -2 to -4.5 . They found that gravothermal oscillations appeared when the total mass of the system (M_{tot}) was of order $8 \times 10^4 M_{\odot}$ (see Murphy et al 1990, Figure 6) and that the critical value of M_{tot} increased with decreasing power law exponent. They suggested that the appearance of oscillations depends on the number of heavier stars. However, this leads to the issue that in a multi-component system it is not clear what the definition of a heavy star should be (this point is discussed in Section 2).

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The main aim of the present paper is to provide a theoretical understanding of the onset of gravothermal oscillations in multi-component systems. As this present paper follows on from the research of Breen & Heggie (2012) it is worthwhile attempting to extend the concepts developed in that paper to the multi-component case. Although two-component systems may be realistic approximations of multi-component systems (Kim & Lee 1997), it is best to have a better understanding of gravothermal oscillations in multi-component systems as real globular clusters contain a continuous mass spectrum. What is of particular importance is the effect of varying the maximum stellar mass (m_{max}) on the onset of instability as this was not studied by Murphy et al (1990).

The rest of this paper is structured as follows. In Section 2, we state the results concerning gravothermal oscillations in gaseous models. This section also contains subsections on the Goodman stability parameter and a variant which used a modified relaxation timescale. This is followed by Section 3 in which the results of N-body simulations are given. Finally, Section 4 consists of the conclusion and discussion.

2 CRITICAL VALUE OF N

2.1 Results of gaseous models

In all cases, the initial conditions used were realisations of the Plummer model (Plummer 1911; Heggie & Hut 2003). The initial velocity dispersion of all components and the initial ratio of density of all components were equal at all locations. The initial conditions were constructed in order to approximate a continuous power law IMF with different exponents $\alpha = -0.0, -0.65, -1.3, -1.65, -2.00$ and -2.35 . The multi-component gas code SPEDI¹ was used for all gaseous models in the present paper. The power law IMFs were approximated by dividing the complete mass range into equal logarithmic steps. Alternative methods of discretization were also tried for certain cases in order to confirm the validity of the results, such as approximating the IMF using equal total masses in each of the components. The ranges of stellar masses (m_{max}, m_{min}) used in this paper are (1.0, 0.1), (2.0, 0.1) and (3.0, 0.1). The reason why higher values of m_{max} were not used is that it is customary to suppose that a cluster would be largely depleted in heavier stars by the time gravothermal oscillations manifest (Kim, Lee & Goodman 1998). In Sec 4, however, we briefly discuss a possible exception.

The critical value of N (N_{crit}) at which oscillations in the central density (ρ_c) first appeared (as N increased) was determined (correct to ten percent). The obtained values of N_{crit} in units of 10^4 are given in Table 1.

¹ SPEDI is a multi-component gas code which was initially based on a formulation by Louis & Spurzem (1991) and was subsequently further developed by Spurzem & Takahashi (1995). Further information regarding SPEDI is available at <http://www.ari.uni-heidelberg.de/gaseous-model/>.

Table 1. Critical value of N (N_{crit}) in units of 10^4 . The values of N_{crit} in brackets were obtained using 5-component models, while all other values were obtained using 10-component models. The value of N_{ef} for the case $\alpha = -2.35$ and extreme masses (3, 0.1) could not be obtained with a 10-component model due to numerical difficulties.

$(m_{max}, m_{min}) \backslash \alpha$	0	-0.65	-1.3	-1.65	-2.0	-2.35
(3, 0.1)	2.0	2.8	5.2	8.0	12.0 (10.0)	– (15.7)
(2, 0.1)	2.0	2.6	4.5	7.0	8.0	10.0 (10.0)
(1, 0.1)	2.0	2.5	3.5	4.5	6.0	8.0

Table 2. Critical value of M (M_{crit}) in units of 10^4

$(m_{max}, m_{min}) \backslash \alpha$	0	-0.65	-1.3	-1.65	-2.00	-2.35
(3, 0.1)	3.1	3.1	3.4	3.8	4.2	(4.3)
(2, 0.1)	2.1	2.0	2.3	2.8	2.5	2.6
(1, 0.1)	1.1	1.1	1.2	1.3	1.5	1.8

2.2 Interpretation of results

Guided by the results of Murphy et al (1990), we first consider the values of M_{tot} at N_{crit} (M_{crit}). The values of M_{crit} , for the models considered in the present paper, are given in Table 2. The values of M_{crit} in Table 2 are approximately the same for fixed m_{max} and the values vary much less with α than N_{crit} . Thus the conclusion of Murphy et al (1990), who considered only evolved IMFs with fixed m_{max} , appears also to apply to pure power law IMFs with fixed m_{max} . What has been added in the study in the present paper is that the value of M_{crit} also has a strong dependence on m_{max} .

We can compare the dependence on m_{max} in Table 2 with the results of the two-component models of Breen & Heggie (2012) if we fix the stellar mass of the light component in that earlier paper. This is done in Appendix A (see Table A2). In Table A2 there is a clear trend of increasing M_{crit} with increased stellar mass of the heavy component for fixed total mass ratio (i.e. moving up through one column of Table A2). Therefore the trend of increasing M_{crit} with increasing stellar mass range (or stellar mass ratio), for fixed m_{min} , seems to be a common feature of multi-component systems. It is also worth noting that for two-component systems with fixed stellar mass ratio (see Appendix A Table A2 where we consider values along a given row) there is a trend of increasing M_{crit} with decreasing total mass ratio. As decreasing total mass ratio for two-component systems is the analogue of decreasing α , these systems have the same stability trends as the multi-component systems in Table 2. Now we will attempt to extend the interpretation of Breen & Heggie (2012) from two-component to multi-component models in a way which also accounts for the dependence on m_{max} .

Breen & Heggie (2012) first argued that for the two-component case, the dynamics of the system were dominated by the heavier component. The reasoning behind this emphasis on N_2 is that the heavier component concentrates within the central region where it behaves like a one-component system deep in the potential well of the low mass stars, and it can exhibit gravothermal instability

like the central part of a one-component system². Breen & Heggie (2012) showed that N_2 , the number of heavy stars, does indeed provide an approximate criterion for the onset of instability, and found the critical value of N_2 to be of order 2000. For the case of multi-component models, however, it is unclear if and how the system can be divided into a heavy and a light component. Nonetheless, it may still be expected that the heavier stars may be more important to the dynamics of the system and to the onset of gravothermal oscillations.

Breen & Heggie (2012) then gave an alternative measure of the importance of the heavy component, in terms of what they called the “effective” number of heavy stars, and this is a concept that is more readily adapted to the case of several components. Breen & Heggie (2012) argued that, as the light component acts as a kind of container for the heavy component, it was the overall mass of this container (i.e. the total mass in the light component) that was the important factor and not the stellar mass of the light component. Therefore it could be replaced by an equal mass of stars of the massive component, giving rise to the idea of the effective number of stars N_{ef} . This was defined as follows

$$N_{ef} = \frac{M_1 + M_2}{m_2}. \quad (1)$$

They also defined a modified half-mass relaxation time scale ($t_{ef,rh}$) by using N_{ef} in place of N in the standard formula for the half-mass relaxation time. They used this effective relaxation time to modify and improve upon a stability criterion suggested by Goodman (1993); see Section 2.3 of the present paper. It is worth pointing out that N_{ef} itself can be used as an approximate stability condition for the two-component models of Breen & Heggie (2012) (see Appendix A). It is this form of condition for the stability of two-component systems which we shall attempt to generalise to multi-component systems.

Again it is not immediately clear what the most appropriate extension of N_{ef} to the multi-component case should be, as the appropriate definition of heavy stars is not clear. However, the result is much less sensitive to our choice than the number of heavy stars, N_2 . One simple approximate way to define a heavy star in this context is simply any star with a stellar mass of $\approx m_{max}$. This would lead to the definition of N_{ef} as $\frac{M_{tot}}{m_{max}}$ for the multi-component model and no change in the definition for a two-component model (equation (1)). This value is given in Table 3, for the multi-component models considered in this paper. We find that there is much less variation among the critical values of N_{ef} than for M_{crit} , especially for varying m_{max} . Nevertheless, the same trend of increasing M_{crit} with decreasing α as seen in Table 2 is still present in Table 3 in the form of increasing N_{ef} .

Now we will discuss a possible explanation for the increase in the critical value of N_{ef} with decreasing α in Table

Table 3. Critical value of N_{ef} in units of 10^4 .

$(m_{max}, m_{min}) \backslash \alpha$	0	-0.65	-1.3	-1.65	-2.00	-2.35
(3,0.1)	1.1	1.0	1.1	1.3	1.4	(1.4)
(2,0.1)	1.1	1.0	1.2	1.4	1.3	1.3
(1,0.1)	1.1	1.1	1.2	1.3	1.5	1.8

3. The idea behind N_{ef} is that a multi-component system behaves in approximately the same way as a single-component system consisting of N_{ef} stars of stellar mass m_{max} . As systems with higher α have more stars with stellar mass $\approx m_{max}$ than systems with lower α , we may expect that the approximation with the one-component system is better for higher α than for lower α . Therefore, it is not surprising that in Table 3 it is the systems with $\alpha = 0$ that have the closest critical values of N_{ef} to the critical value of N for a one-component system, i.e. about 7000.

We can take our discussion further by considering systems with fixed N_{ef} . For systems with fixed N_{ef} , as α decreases there is an increasing number of light stars (stars with stellar mass not $\approx m_{max}$). As the number of light stars increases and the number of heavy stars decreases the two-body relaxation time increases, as it becomes increasingly dominated by the light component. According to Hénon’s principle (Hénon 1975) the rate of energy generation in the system is regulated by two-body relaxation, and therefore there is a lower rate of energy generation as α decreases. Regardless of the value of α , the average mass in the core is approximately m_{max} , and the lower rate of energy generation can be met by a core of lower density. Thus for lower α there is a smaller density contrast between the core and the mean density in the system. This would imply that the stability would increase with decreasing α , as we indeed see.

2.3 Goodman stability parameter

Goodman (1993) suggested the use of the quantity

$$\epsilon \equiv \frac{E_{tot}/t_{rh}}{E_c/t_{rc}} \quad (2)$$

as a stability indicator, where $\log_{10} \epsilon \sim -2$ is the stability limit below which the cluster becomes unstable. Here E_{tot} is the total energy, E_c is the energy of the core, t_{rc} is the core relaxation time and t_{rh} is the half mass relaxation time. A condition of this type has been supported for two-component models by Kim, Lee & Goodman (1998) who studied Spitzer stable models using a Fokker-Planck code and by Breen & Heggie (2012) who studied Spitzer unstable models using a gas code. However, Breen & Heggie (2012) also introduced a modified definition of ϵ (see below) because the definition given in equation (2) was found to yield a critical value which varied with total mass ratio and stellar mass ratio. Also, the critical value at which the instability appears is somewhat different in Breen & Heggie (2012) from that in Kim, Lee & Goodman (1998).

For the multi-component models studied in the present paper, the values of $\log_{10} \epsilon$ are given in Table 4. The values of $\log_{10} \epsilon$ (based on the original definition, i.e. equation (2)) range from -2.26 to -2.56 and there is a decreasing trend with decreasing α .

² Note that, in this picture, the gravothermal instability of the system is essentially confined to the massive stars; it is in the centrally concentrated massive stars that the temperature inversions occur which drive the expansion phase of the gravothermal oscillations. Around $N = N_{crit}$, the light stars always exhibit a normal temperature gradient, and if their heat capacity is negative this does not lead to instability.

Table 4. Critical value of $\log_{10} \epsilon$

$(m_{max}, m_{min}) \backslash \alpha$	0	-0.65	-1.3	-1.65	-2.00	-2.35
(3,0.1)	-2.26	-2.38	-2.54	-2.61	-2.57	(-2.62)
(2,0.1)	-2.26	-2.32	-2.48	-2.53	-2.58	-2.56
(1,0.1)	-2.26	-2.30	-2.39	-2.40	-2.43	-2.42

In equation (2), t_{rc} and t_{rh} are defined by

$$t_{rc} = \frac{0.34\bar{\sigma}_c^3}{G^2\bar{m}_c\rho_c \ln \Lambda} \quad (3)$$

and

$$t_{rh} = \frac{0.138N^{\frac{1}{2}}r_h^{\frac{3}{2}}}{(G\bar{m})^{\frac{1}{2}} \ln \Lambda}. \quad (4)$$

where \bar{m}_c and $\bar{\sigma}_c$ are the mass density weighted averages over all the components in the core. However, the definition of t_{rh} does not take into account the mass spectrum. Breen & Heggie (2012) have shown that for two-component models, modifying the definition of t_{rh} to take into account the mass spectrum leads to an improved stability condition. As has already been shown in Section 2.1, we can construct a value N_{ef} which provides an approximate stability condition. Now we will use this value to define a new half mass relaxation time defined as

$$t_{ef,rh} = \frac{0.138N_{ef}^{\frac{1}{2}}r_h^{\frac{3}{2}}}{(Gm_{max})^{\frac{1}{2}} \ln \Lambda}. \quad (5)$$

We use this in place of t_{rh} in equation (2) to define the new stability parameter ϵ_2 as in Breen & Heggie (2012). The values of this parameter are given in Table 5. The variation of $\log_{10} \epsilon_2$ in Table 5 is of comparable magnitude to the variation of $\log_{10} \epsilon$ in Table 4, but the values in Table 5 are more consistent with that of a one-component model ($\log_{10} \epsilon = \log_{10} \epsilon_2 = -2$) than those in Table 4.

For the one-component model the definitions of ϵ and ϵ_2 are identical. Also, it is worth noting that, for two-component systems with extremely small amounts of the heavy component (relative to M_{tot}), $t_{ef,rh}$ as defined in equation (5) is not a suitable approximation to the relaxation time. This is the case for the models considered by Kim, Lee & Goodman (1998), and so the extension to ϵ_2 would not have been necessary or useful in the context of their paper.

The logarithm of the Goodman stability parameter (or our somewhat more consistent modified version) seems to have a particular value approximately -2 at the stability boundary (Kim, Lee & Goodman (1998), Breen & Heggie (2012) and the present paper). However, it is not known if ϵ (or ϵ_2) can be predicted for a particular IMF without carrying out numerical simulations, which limits its usefulness. In contrast N_{ef} has the advantage that it can be easily calculated before carrying out numerical simulations, and also provides an approximate indication of the stability boundary.

3 DIRECT N-BODY SIMULATIONS

In order to validate the results obtained from the gas code SPEDI a series of N -body runs were carried out. The direct

Table 5. Critical value of $\log_{10} \epsilon_2$

$(m_{max}, m_{min}) \backslash \alpha$	0	-0.65	-1.3	-1.65	-2.00	-2.35
(3,0.1)	-2.06	-2.02	-1.99	-1.94	-1.75	(-1.73)
(2,0.1)	-2.04	-1.99	-2.00	-1.95	-1.75	-1.81
(1,0.1)	-2.05	-2.01	-2.01	-1.96	-1.94	-1.87

N -body simulations in the present paper were conducted using the NBODY6 code (Aarseth 2003; Nitadori & Aarseth 2012). As the IMF's with $\alpha = 0$ in Table 1 have the lowest values of N_{crit} (2.0×10^4), for this value of α N -body runs were carried out for the mass ranges (1,0.1) and (2,0.1). The case with parameters $\alpha = -1.3$ and $(m_{max}, m_{min}) = (1.0, 0.1)$ was also chosen because it has a higher value of N_{crit} (3.5×10^4) than for $\alpha = 0$, although the value is still low enough to make it suitable for direct N -body simulations.

For the case of $\alpha = 0.0$ the values of N_{crit} are the same (see Table 1) regardless of the stellar mass range. For the two mass ranges chosen, there were no signs of gravothermal behaviour in the N -body runs with $N = 8k$ or $N = 16k$. The first clear sign of gravothermal behaviour occurs with $N = 32k$ for both chosen mass ranges. For the stellar mass range (1.0, 0.1), the three panels of Fig. 1 show, respectively, (i) the evolution of the core radius r_c , (ii) an example of a single cycle of gravothermal oscillation in the post-collapse evolution, and (iii) evidence of the gravothermal nature of the oscillation for the $32k$ run. The same graphs for the $32k$ run with stellar mass range (2.0, 0.1) are given in Fig. 2.

For the case $\alpha = -1.3$ and stellar mass range (1.0, 0.1) the value of N_{crit} is 3.5×10^4 (see Table 1). No gravothermal behaviour was seen in the N -body runs with $N = 8k$, $12k$ and $32k$ for this set of conditions. The first signs of gravothermal behaviour occurred in the $64k$ run as would be expected from the above value of N_{crit} obtained from SPEDI. The same three graphs shown for both the $\alpha = 0.0$ cases (see previous paragraph) are plotted for the $64k$ run in Fig. 3.

In the graphs of r_c for all cases (see Fig. 1 top, Fig. 2 top and Fig. 3 top), behaviour can be seen which is qualitatively similar to gravothermal oscillations (see Makino (1996), Takahashi & Inagaki (1995), Heggie & Giersz (2009) and Breen & Heggie (2012)). For the case of $\alpha = 0.0$ with the mass range (1.0, 0.1) (Fig. 1 top) one oscillation can be seen between 3830 and 4570, and another between 5970 and 6560. During each of these oscillations r_c changes by more than a factor of 10. Similarly for the case of $\alpha = 0.0$ with the mass range (2.0, 0.1) (Fig. 2 top) an oscillation in r_c can be seen between 6800 and 7950, and part of an oscillation can also be observed after 8610. In this model the change in r_c is about a factor of 10. Finally for the case of $\alpha = -1.3$ with the mass range (1.0, 0.1) (Fig. 3 top) an oscillation in r_c can be seen between 4800 and 5600, and part of an oscillation can also be observed after 7400. The change in r_c is about a factor of 10, which is similar to the change in r_c for the $\alpha = -1.3$ runs.

Now we consider the physical nature of these oscillations, which could in principle be driven by sustained binary activity or by gravothermal behaviour. A sign of gravothermal behaviour is that the binding energy of the binaries remains roughly constant during times of expansion (McMil-

lan & Engle 1996). This is because the expansion phase of a gravothermal oscillation should be driven by the core absorbing heat from the rest of the cluster rather than by energy generation. At core bounce, where ρ_c reaches a local maximum, there is an increase in binary activity, and enough energy is produced to halt and reverse the collapse. This behaviour is particularly clear in Fig. 3 (middle) where there is an initial increase in relative binding energy of the binaries coinciding with core bounce and the initial expansion. A binary escapes, and then there is a period of expansion during which the relative binding energy of the binaries remains nearly constant (from 4850 to 5000). Towards the end of the oscillation there is renewed binary activity corresponding to the next core bounce. There is also binary activity at other times during the oscillation, but it has no discernible effect on the evolution of r_c . Fig. 1 (middle) is also a good example of gravothermal behaviour. Mild binary activity continues after core bounce (from $t = 3860$ to $t = 3930$) but expansion continues thereafter for a period. However in Fig. 2 evidence of gravothermal behaviour is more ambiguous. We will discuss the case of Fig. 2 in detail in the last paragraph of this section.

The cycles of ρ_c vs the core velocity dispersion v_c^2 , as seen in Fig. 1 (bottom), Fig. 2 (bottom) and Fig. 3 (bottom), are believed to be a sign of gravothermal behaviour (Makino 1996). During these cycles, the temperature is lower during the expansion where heat is absorbed and higher during the collapse where heat is released. The velocity dispersion in Fig 2 (bottom) and Fig. 3 (bottom) has been smoothed to make the cycle clearer. These cycles are similar to the cycles found by Makino (1996) for one-component models and by Breen & Heggie (2012) for two-component models.

The gravothermal nature of the behaviour is clearer in Fig. 1 (where $(m_{max}, m_{min}) = (1.0, 0.1)$) than in Fig. 2 (where $(m_{max}, m_{min}) = (2.0, 0.1)$). This is perhaps surprising as both cases have the same value of N_{crit} as found with SPEDI. We will now discuss a number of possible reasons for this apparent difference in behaviour. Firstly, as the values of N_{crit} are only correct to 10%, it is possible that in reality the values could differ by up to 4×10^3 . Secondly, another issue is that in the gas model the mass function is discretised, resulting in a difference between the mass of the heavy component (m_{10}) and m_{max} . m_{10} is about 14% percent less than m_{max} for the stellar mass range (2.0, 0.1) and 11% for (1.0, 0.1). It is argued in Appendix A that the stability of a system will increase if the average stellar mass inside the core is increased (while keeping the stellar masses outside the core approximately the same). This would imply that in both cases the values of N_{crit} found with the 10-component models are underestimates for the onset of instability, and that the true value of N_{crit} for (2.0, 0.1) is slightly higher than for (1.0, 0.1). Finally, the fact that the gravothermal nature of the behaviour is clearer in one run might simply be a stochastic effect.

For the purposes of this paper, we have not considered the evolution of multi-component systems in the regime $N < N_{crit}$. In Spitzer-unstable cases, there is no reason to doubt that this is characterised by mass-segregation, followed by post-collapse expansion powered by binary evolution as in the much smaller N-body models considered long ago by van Albada (1967), Aarseth (1968) and many more since.

4 SUMMARY AND DISCUSSION

The focus of this paper has been on the conditions for the onset of gravothermal oscillations in multi-component systems. We have investigated power law IMFs with different exponents and three different stellar mass ranges (3.0, 0.1), (2.0, 0.1) and (1.0, 0.1). A multi-component gas code has been used to obtain the values of N_{crit} . In order to verify the validity of the results direct N -body runs were carried out on appropriately chosen cases. The values of N_{crit} found ranged from 2×10^4 to 10^5 , and varied with α and the stellar mass range.

Motivated by Murphy et al (1990), who found that the total mass of the systems they studied could be used as an approximate stability condition, the value of M_{crit} (the total mass of the system at N_{crit}) for each system was calculated (see Table 2). While for a fixed mass range M_{crit} does provide an approximate stability condition, the value of M_{crit} varied by roughly a factor m_{max} . M_{crit} can also be used as an approximate stability condition for the two-component models of Breen & Heggie (2012) so long as the stellar mass ratio is fixed (see Appendix A).

In order to find a more general stability condition we applied an extension of an idea first employed in Breen & Heggie (2012). They used a quantity called the effective particle number (N_{ef}). The value was useful because the two-component system that was being considered was expected to behave in roughly the same manner as a one-component system with N_{ef} stars. In the present paper this idea has been extended to multi-component systems. The values of N_{ef} for the multi-component models in this paper are given in Table 3. The variation in Table 3 is significantly less than that in either Table 1 or Table 2. A stability condition of $N_{ef} \sim 10^4$ covers most of the values of Table 3 and indeed the two-component models of Breen & Heggie (2012) (see Table A3).

The Goodman Stability Parameter was also tested for the multi-component case (see Table 4). The critical values in Table 4 were found to be lower than the value for a one-component model ($\log_{10} \epsilon = -2$) and also varied with α and, to a much lesser extent, with m_{max} . By modifying the Goodman Stability Parameter using a slightly different definition for the half-mass relaxation time (based on the effective particle number) a critical value was found which was more consistent with the critical value for a one-component model (see Table 5).

Goodman (1987) used a gas model to find the value of N_{crit} (= 7000) for a single component system. Technically what he showed was that steady post-collapse expansion was possible in a gas model for all N , but that it was unstable for $N > 7000$. While the gas model used by Goodman (1987) is similar in form to the model used here and in Breen & Heggie (2012), there are two notable differences. Firstly Goodman (1987) used a larger energy generation rate than the one used here. Secondly, the parameter of the coulomb logarithm that was used was $\lambda = 0.4$. A value of $\lambda = 0.4$ (Spitzer 1987) was a reasonable choice at the time, but it has since been shown that $\lambda = 0.11$ is a better choice for a single-component model (Giersz & Heggie 1994). (For multi-component models the value of $\lambda = 0.02$ was found to provide a good fit (Giersz & Heggie 1996)). These two differences affect the stability in opposite ways: by arguments similar to those given in the

last two paragraphs in Appendix A, a larger energy generation rate will increase stability, whereas a larger value of λ tends to reduce stability. For example for $N = 7000$ with $\lambda = 0.4$ the increase in the relaxation rate is 20% compared with $\lambda = 0.11$.

In the present paper, we have made the assumption that multi-component systems will be depleted in stars with stellar mass greater than $3M_{\odot}$. This neglects the possibility of systems containing a population of stellar mass Black Holes, which would require a value of m_{max} about an order of magnitude greater than what is considered here. These systems are outside the parameter space studied by Breen & Heggie (2012) and Kim, Lee & Goodman (1998), as the total mass ratio is lower than the range considered by Breen & Heggie (2012) and the stellar mass ratio is higher than the values considered by Kim, Lee & Goodman (1998). The onset of gravothermal oscillations and the more general evolution of systems containing a population of stellar mass black holes are the topics of the next paper in this series.

To conclude, a stability condition of $N_{ef} \sim 10^4$ does apply to the multi-component systems in this paper and the two-component systems of Breen & Heggie (2012). This condition is expected to apply to any multi-component system provided that there is a sufficient number of stars with stellar mass $\sim m_{max}$.

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APPENDIX A: THE TWO-COMPONENT CASE REVISITED

The purpose of this appendix is to reconsider the results of Breen & Heggie (2012) in terms of the effective particle number N_{ef} defined in equation 1. Breen & Heggie (2012) investigated gravothermal oscillation in a range of two-component models, specified by the stellar mass ratio $\frac{m_2}{m_1}$ and total mass ratio $\frac{M_2}{M_1}$, where m_2 (m_1) is the stellar mass of the heavy (light) component and M_2 (M_1) is the total mass of the heavy (light) component. For reference the values of N_{crit} for these models are given in Table A1, which is similar to Table 2 in Breen & Heggie (2012). The difference is that the data have been rearranged to compare more closely to the arrangement in the present paper. Thus the columns in Table A1 are arranged in order of decreasing $\frac{M_2}{M_1}$ as this is the analog for two components of decreasing α .

Following the approach in the main part of this paper, we will firstly consider the values of M_{crit} . In order to consider M_{crit} we need to specify the mass unit, and to make the results comparable with the multi-component models in the main part of the present paper m_1 has been fixed at $0.1M_{\odot}$. The values of M_{crit} for the two-component models in Table A1 are given in Table A2. For comparison the value of M_{crit} for a one-component model would be 0.07×10^4

Table A1. Critical value of N (N_{crit}) in units of 10^4

$\frac{m_2}{m_1} \setminus \frac{M_2}{M_1}$	1.0	0.5	0.4	0.3	0.2	0.1
50	18	30	33	42	55	100
20	8.5	13	15	18	22	36
10	5.0	7.2	8.2	10	12	22
5	2.8	4.0	4.6	5.4	7.0	10
4	2.4	3.5	3.8	4.6	5.5	8.5
3	2.0	2.8	3.2	3.6	4.4	6.0
2	1.7	2.2	2.3	2.6	3.0	3.8

Table A2. Critical value of M_{crit} in units of $10^4 M_\odot$. The value of m_1 is fixed at $0.1 M_\odot$. For reference the value of M_{crit} is $0.07 \times 10^4 M_\odot$ for $\frac{M_2}{M_1} = 0$, which is obtained from the result of Goodman (1987) for a one-component cluster with $m_1 = 0.1 M_\odot$.

$\frac{m_2}{m_1} \setminus \frac{M_2}{M_1}$	1.0	0.5	0.4	0.3	0.2	0.1
50	3.529	4.455	4.583	5.427	6.574	10.978
20	1.619	1.902	2.059	2.305	2.614	3.940
10	0.909	1.029	1.104	1.262	1.412	2.396
5	0.467	0.545	0.596	0.662	0.808	1.078
4	0.384	0.467	0.484	0.556	0.629	0.912
3	0.300	0.360	0.395	0.425	0.495	0.639
2	0.227	0.264	0.268	0.294	0.327	0.398

(using $m = 0.1 M_\odot$). This is significantly lower than any of the values in Table A2³. For fixed values of $\frac{m_2}{m_1}$ the value of M_{crit} varies by factors of up to ≈ 3 between $\frac{M_2}{M_1} = 1$ and 0.1 . Therefore for fixed $\frac{m_2}{m_1}$, M_{crit} does provide a rough stability condition. However for fixed $\frac{M_2}{M_1}$, the variation in M_{crit} is a factor of $\approx 15 - 30$. The variation of M_{crit} with varying m_2 resembles the variation of M_{crit} with varying m_{max} in Table 2.

We will now consider the values of N_{ef} for the two-component systems; these are given in Table A3. For comparison the critical value of N_{ef} for a one-component model is the same as its value of N_{crit} , which is 0.7×10^4 . The values in Table A3 vary much less than those of M_{crit} in Table A2, although, as pointed out in Section 2.1, N_{ef} can be interpreted as a measure of the total mass of the system in units of m_2 . A stability condition of $N_{ef} \sim 10^4$ or slightly more covers, within a factor 2 at most, the values of Table A3 and indeed Table 3.

³ In Table A2 for fixed $\frac{m_2}{m_1}$, M_{crit} increases with decreasing $\frac{M_2}{M_1}$. Given that M_{crit} for $\frac{M_2}{M_1} = 0$ is significantly lower than any of the values in Table A1, one may wonder if that increasing trend observed in Table A1 continues below $\frac{M_2}{M_1} = 0.1$. This is a topic considered in detail in the next paper of this series.

Table A3. Critical value of N_{ef} in units of 10^4

$\frac{m_2}{m_1} \setminus \frac{M_2}{M_1}$	1.0	0.5	0.4	0.3	0.2	0.1
50	0.71	0.89	0.91	1.09	1.31	2.20
20	0.81	0.95	1.02	1.15	1.31	1.97
10	0.91	1.03	1.10	1.26	1.42	2.40
5	0.93	1.09	1.19	1.32	1.62	2.16
4	0.96	1.17	1.21	1.39	1.57	2.28
3	1.00	1.20	1.32	1.42	1.65	2.13
2	1.13	1.32	1.34	1.47	1.64	1.99

All of the trends in Table A3 may be understood if we consider the reasoning behind the use of N_{ef} as an approximate stability condition. The basic idea is that the multi-component system in question evolves in a similar way to a one-component system of N_{ef} stars with stellar mass m_2 . This requires that the half mass relaxation timescale of the multi-component system is similar to that of the one-component system with which we are comparing it. We assume this to be true if the heavy component amounts to a significant fraction of the total mass within the half-mass radius r_h , which is certainly not the case as $\frac{M_2}{M_1}$ tends to 0, i.e. on the extreme right of Table A3.

We now consider with more care how the two-component system actually differs from the corresponding one-component system as the parameters $\frac{M_2}{M_1}$ and $\frac{m_2}{m_1}$ are varied. For fixed $\frac{m_2}{m_1}$, as $\frac{M_2}{M_1}$ decreases the relaxation process is increasingly dominated by the light stars. This leads to the system behaving more like a one-component system of $\frac{M_{tot}}{m_1}$ stars as opposed to a one-component system of N_{ef} stars, and this increases the half-mass relaxation time. As the rate of two body relaxation becomes slower the core becomes larger (relative to r_h ; see the discussion of Hénon's Principle in Section 2.2) as it can produce the required energy at a lower mass density (as the average stellar mass in the core remains the same, roughly m_2). Because gravothermal instability depends on a high density contrast within the system, it would be expected that stability would increase as $\frac{M_2}{M_1}$ decreases, as can be seen in Table A3.

Now let us consider the case of fixed $\frac{M_2}{M_1}$. If we consider the post collapse evolution of series of systems with fixed M_2 and m_2 , as $\frac{m_2}{m_1}$ decreases the tendency towards mass segregation becomes weaker. Therefore the half mass radius of the heavy component ($r_{h,2}$) is smaller compared to r_h for larger $\frac{m_2}{m_1}$ than for smaller $\frac{m_2}{m_1}$. It follows that the mass density of the heavy component (within $r_{h,2}$) is smaller for smaller $\frac{m_2}{m_1}$ than for larger $\frac{m_2}{m_1}$. The relaxation time of the heavy component within its half-mass radius $r_{h,2}$ ($t_{r_{h,2}}$) decreases with increasing mass density. This leads to the conclusion that the relaxation time within the heavy component increases with decreasing $\frac{m_2}{m_1}$. The energy flux in the heavy component, which we are assuming regulates the rate of energy generation, is of order $\frac{|E_2|}{t_{r_{h,2}}}$ (where E_2 is the energy of the heavy system). Therefore, as $\frac{m_2}{m_1}$ decreases so does the energy flux, which results in a lower rate of energy generation. The lower rate of energy generation leads to a larger core (relative to r_h) as the core can produce the required energy at a lower mass density. Thus it would be expected that stability (as measured by N_{ef}) would increase as $\frac{m_2}{m_1}$ decreases, and this is what is observed in Table A3 for most values of $\frac{M_2}{M_1}$. However, the trend of increasing stability with decreasing $\frac{m_2}{m_1}$ seems to disappear for small $\frac{M_2}{M_1}$. Reasons for this will be discussed in the next paper of this series.

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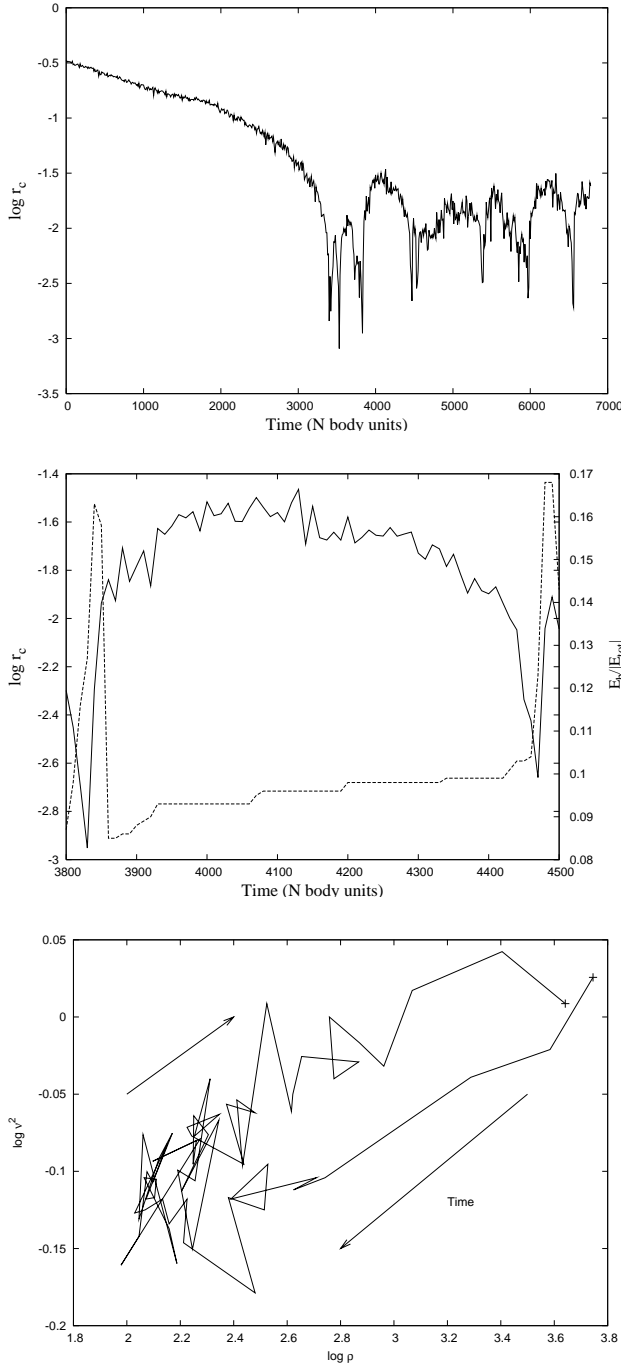


Figure 1. N -body run of a multi-component model with $N = 32k$, $\alpha = 0.0$ and $(m_{\text{max}}, m_{\text{min}}) = (1.0, 0.1)$. **Top:** $\log r_c$ vs time (N body units) over the entire run. **Middle:** $\log r_c$ vs time (N body units) over-plotted with the relative binding energy of binaries. The plot is over the period of a gravothermal oscillation which occurs between 3830 and 4570. **Bottom:** ρ_c vs v_c^2 , showing the gravothermal nature of the cycle over the same time period as the middle plot.

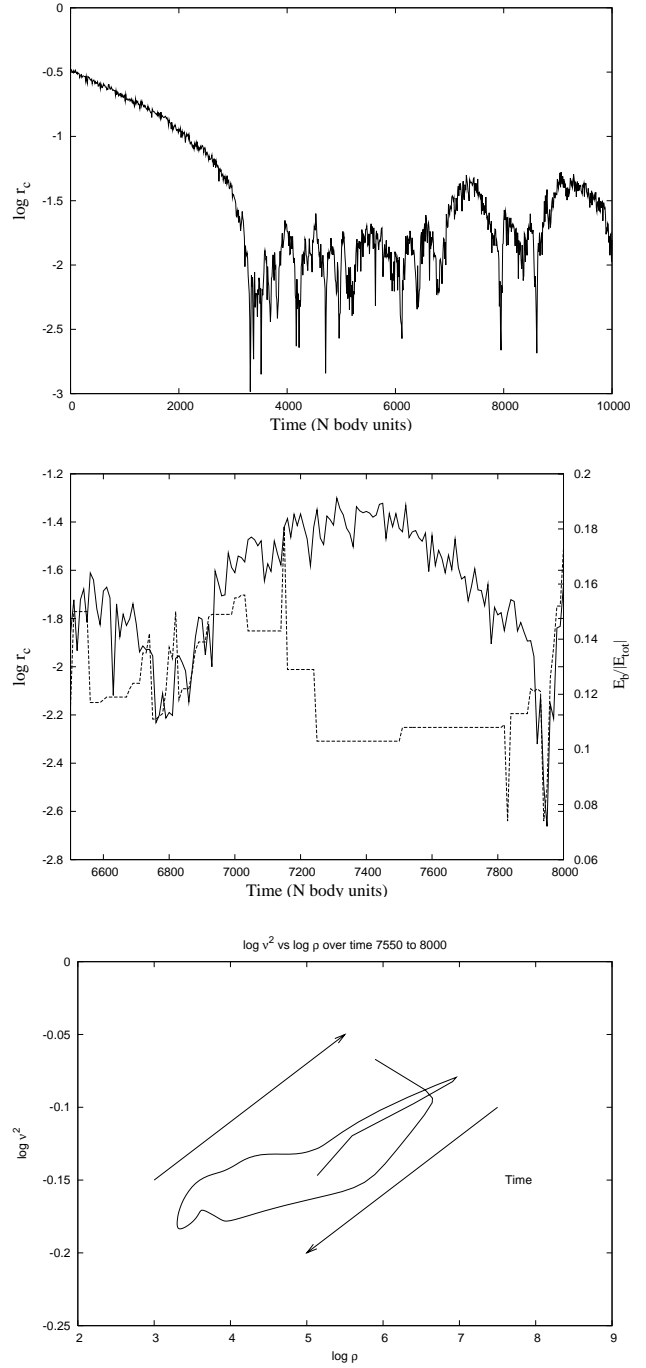


Figure 2. As Fig. 1, with $N = 32k$ and $\alpha = 0.0$, but $(m_{\text{max}}, m_{\text{min}}) = (2.0, 0.1)$. The velocity dispersion has been smoothed to make the cycle clearer.

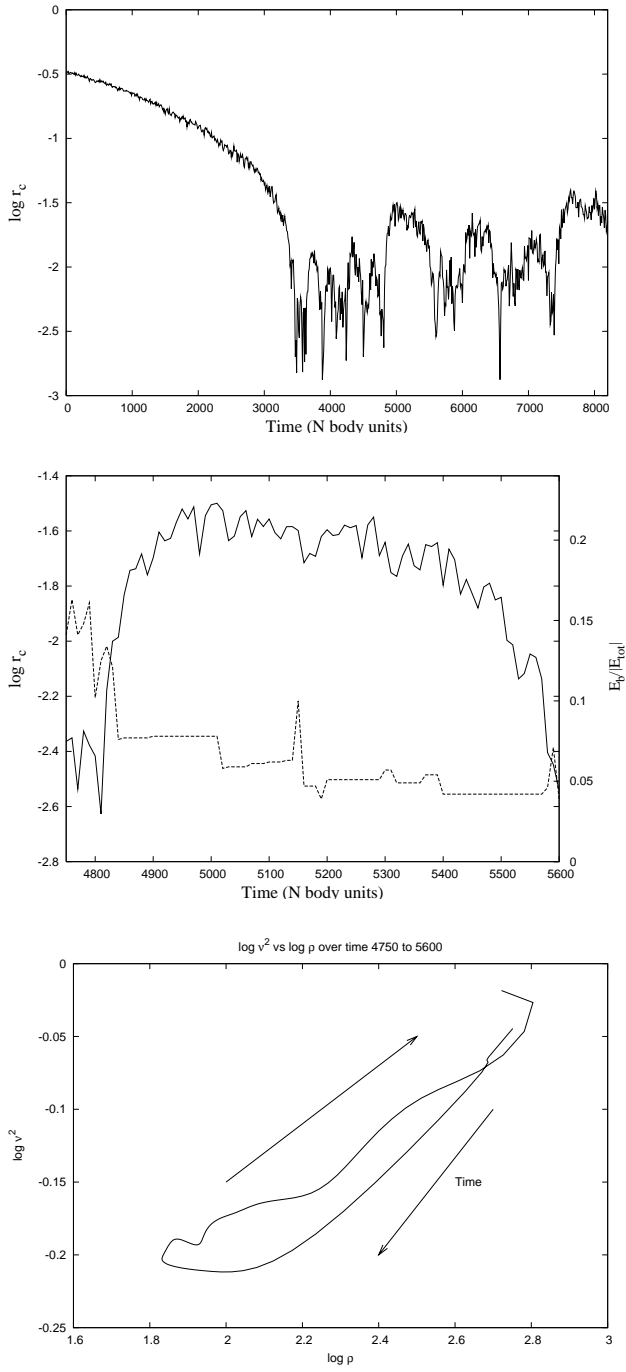


Figure 3. As Fig. 1, with $(m_{max}, m_{min}) = (1.0, 0.1)$, but $N = 64k$ and $\alpha = -1.3$. The velocity dispersion has been smoothed to make the cycle clearer.